Schwarzschild Cavity in a Friedmann Universe

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Current standard models of Big Bang cosmology are based on the fundamental assumption of a homogeneous and isotropic distribution of matter and radiation in our universe. This gives us the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Though valid globally, the universe is clearly inhomogeneous on a local scale: we observe concentrations of matter in some places, and complete vacuum in others. This presentation outlines a proposed reconciliation: the embedding of a Schwarzschild vacuole inside a Friedmann universe. Junction conditions are discussed and the matching is shown by satisfying the Darmois junction conditions. Additionally, we outline the generalization to many vacuoles in an expanding universe: the ΛCDM "Swiss Cheese" Model.

The Problem of Local vs. Global Geometries

Locally, our universe appears to be well-described by the Schwarzschild metric. Originally used to describe the gravitational field around a singular black hole, we may also use it to describe any static, spherically symmetric distribution of matter (Birkhoff's Theorem). Additionally, gravitationally bound systems do not take part in the expansion of the universe — only when gravitational fields become insignificant do particles move with the Hubble flow.

Seminal work by Einstein and Straus[1, 2] showed the existence of a solution for a central body surrounded by a static, spherical vacuum region G embedded in an expanding cosmological background. It should also be worth noting that Oppenheimer and Snyder considered the complementary matching in 1939: a time-dependent Friedmann metric embedded in a Schwarzschild universe[3].

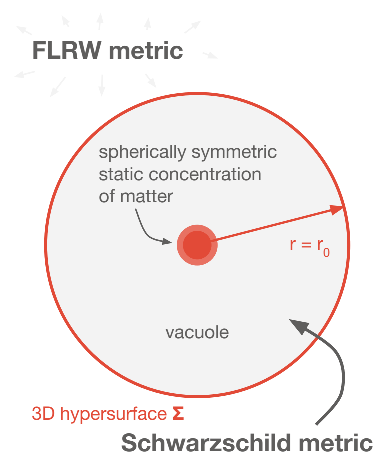

Figure 1: Simple schematic of a Schwarzschild vacuole in a Friedmann universe.

WE BEGIN BY CONSIDERING a Friedmann (dust; $p = 0$) universe in which there exists a single spherically symmetric concentration of matter at $r = 0$, and a sphere of void around it. The FLRW metric is given by[4]:

$$ds^2 = dt^2 - \frac{R^2(t)}{1 - kr^2}dr^2 - R^2(t)r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \quad (1)$$

while the Schwarzschild metric is[1]

$$ds^2 = \left(1 - \frac{2M}{\rho}\right)dT^2 - \left(1 - \frac{2M}{\rho}\right)^{-1}d\rho^2 - \rho^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \quad (2)$$

[1] To avoid confusion we denote the time and radial coordinates in the FLRW metric t, r, and T, ρ in the Schwarzschild metric.

The boundary between the local and global geometries is given by a radially constant hypersurface, Σ. To ensure smooth matching across this boundary, we must consider which junction conditions to satisfy.

Junction Conditions

We consider a general 3D hypersurface, the normal of which can be either spacelike or timelike. Analogously to junction conditions in electromagnetism, gravitational fields also have continuity conditions across a surface that depends on the field's source: in the case of general relativity, it's the energy momentum tensor.

These conditions are derived in the same way in which we do in electromagnetism — perform a "pillbox" integration of the Einstein field equation. First, we introduce Gaussian normal coordinates (for example see chapter 21 of [5]), then perform the integration

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} G^{\alpha}_{\beta} dn = 8\pi \lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} T^{\alpha}_{\beta} dn \tag{3}$$

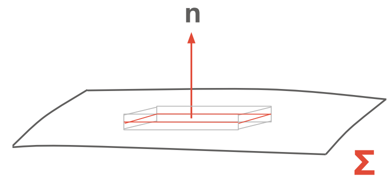

Figure 2: "Pillbox" integration across a hypersurface.

Unsurprisingly, the absence of surface layers imply continuity of the extrinsic and intrinsic metrics. For a discontinuity to be present in the energy-momentum at Σ, Σ must be spacelike. The general junction conditions dictate that momentum flow inside Σ must be contained in Σ, and that while the extrinsic curvature exhibits a delta function 'jump' at the surface, the intrinsic curvature must be continuous across Σ.

The smooth matching of the FLRW and Schwarzschild metrics across Σ is guaranteed if the Darmois junction conditions are satisfied[6]. The first fundamental form concerns the inherited intrinsic metric:

$$\Upsilon_{\alpha\beta} = g_{ij}\frac{dx^i}{du^\alpha}\frac{dx^j}{du^\beta} \tag{4}$$

Note the indicies $i = 1,2,3,4$ and $\alpha = 1,2,3$

The second concerning the 'jump' in the extrinsic metric:

$$\Omega_{\alpha\beta} = (\Gamma^{p}_{ij} n_p - \partial_j n_i)\frac{dx^i}{du^\alpha}\frac{dx^j}{du^\beta} \tag{5}$$

where n is the unit normal to the hypersurface.

Matching the Schwarzschild Cavity

To perform the matching, we must show that the fundamental forms $\Upsilon_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ are the same in the FLRW *and* Schwarzschild metrics. We first assume $\Upsilon_{F\alpha\beta} = \Upsilon_{S\alpha\beta}$, and show from that $\Omega_{F\alpha\beta} = \Omega_{S\alpha\beta}$.

Consider a hypersurface defined by a function in the FLRW metric

$$f_F(x^i) = r - r_0 = 0 \tag{6}$$

Since (4) is satisfied, we focus on condition (5). Since the surface is defined in the FLRW metric, $\Omega_{F\alpha\beta}$ was simple to find:

$$\Omega_{F\alpha\beta} = -\frac{1}{2}|g_{22}|^{-1/2}\partial_2 g_{\alpha\beta} \tag{7}$$

The work here is in finding the components of the normal vector in the Schwarzschild metric, n_{Si}. Trying to get n_{Si} as a function of x^i_S was also slightly tricky. Eventually we show that

$$n_{Si} = \left(\epsilon \frac{d\rho}{du}, -\epsilon \frac{dT}{du}, 0, 0\right) \qquad \epsilon = \pm 1 \tag{8}$$

and differentiating the orthogonality relation[2] with respect to u^α gives us the needed relation:

[2] This is just saying that the normal is orthogonal to Σ, i.e., $n_{Si} \frac{\partial x^i_S}{\partial u^\alpha} = 0$

$$\partial_j n_{Si} \frac{\partial x^i_S}{\partial u^\alpha} \frac{\partial x^j_S}{\partial u^\beta} = -n_{Si} \frac{\partial^2 x^i_S}{\partial u^\alpha \partial u^\beta} \tag{9}$$

This implies that $\Omega_{F\alpha\beta} = \Omega_{S\alpha\beta} = 0, \quad \forall \alpha \neq \beta$. The remaining diagonal components result in three differential equations, which can be shown to be equivalent with a few lines of working.

We must not forget that we also need to verify that the pressure across the boundary is continuous. However, this is trivial as the pressure is zero in Schwarzschild spacetime, and we've matched it up to a Friedmann (dust) universe.

The ΛCDM Swiss Cheese Model

Though we've performed a matching for a single Schwarzschild cavity in a Friedmann universe, nothing's stopping us from adding more. Models with more cavities are, adorably, coined 'Swiss-cheese' models.

Moreover, there's also nothing stopping us from adding a cosmological constant, Λ. We know from observations that our universe has a nonzero Λ, so it makes sense to do so. The current standard model of Big Bang cosmology – the ΛCDM model – is a homogeneous, isotropic model with an FLRW metric. This can be matched up with Schwarzschild-de Sitter, or Kottler metric, which is a stationary, spherically symmetric metric with a nonzero cosmological constant[7].

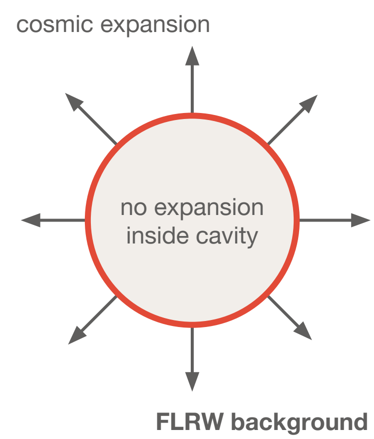

Figure 3: The ΛCDM Swiss-Cheese Model.

References

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